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CLASSICAL MECHANICS

DSE 1

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For the course content, see the syllabus.

[5th Semester] [Classical Dynamics]

1. Lagrangian and Hamiltonian Formulation:

Newtonian mechanics: Based on Forces.

$m\ddot{\vec{r}} = ma - F$.

Lagrangian and Hamiltonian: based on Energy.

Lagrange: $\boxed{L = T - V}$

T: kinetic en, V: potential en.

Hamilton: $\boxed{H = T + V}$ $\quad$ $q_i \to i \to 1$ to n $(q_1, q_2, \dots q_n)$.

No. of degrees of freedom: $n = 3N - k$

N: no. of particles, k: no. of constraints

$\boxed{L = L(q_i, \dot{q}_i, t)}$, $\boxed{H = H(q_i, p_i, t)}$.

q_i: generalised position, $\dot{q}_i$: generalised velocity

p_i: generalised momentum.

classification of constraints:

1. Holonomic: $f(r_1, r_2, r_3, \dots, t) = 0$
   Non-holonomic: Not Holonomic.
   Also fn of gen. vel.
2. Scleronomic: Time independent
   $L = L(q_i, \dot{q}_i)$.
   Rheonomic: Time dependent.
3. conservative: Total mechanical energy of the system is conserved.
   Dissipative: En. not cons. Forces are doing some work. (of const.).
4. Unilateral, bilateral.

Example of constraints:

1. Rigid body: $|\vec{r_i} - \vec{r_k}| = c_{ik} = \text{const}$.
   (Holonomic). (scleronomic). (bilateral) (conservative)
2. Deformable body: $|\vec{r_i} - \vec{r_j}| = f(t)$. $(\forall\ i, j)$.
   Holonomic, rheonomic, bilateral, Dissipative.
   $(dW \neq 0)$.

3. Simple pendulum with rigid support:

position of bob: $|\vec{r}|^2 = x^2+y^2+z^2 = l^2$.

Holonomic, scleronomic, cons., bilateral.

4. Pendulum with variable length:

$l = l(t)$, $|\vec{r}|^2 \to f^n$ of t

Holonomic, Rheonomic, bilateral, Dissipative.

5. Bead is sliding on a wire rotating uniformly about z-axis.

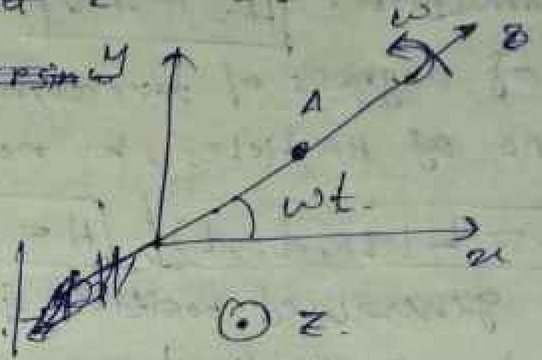
$x = r\cos\omega t$, ~~$y = r\sin$~~

$y = r\sin\omega t$.

~~Holonomic~~

$\tan\omega t = \frac{y}{x}$

$\Rightarrow y - x\tan\omega t = 0$.



Holonomic, Rheonomic

Principle of virtual work:

Let $\vec{F}_{ik}$ denote the force on i^{th} particle due to k^{th} particle in a rigid body. So, work done in a displacement $d\vec{r}_i$ of i^{th} particle is,

$$dW_i = \sum_k \vec{F}_{ik} \cdot d\vec{r}_i \quad -(1)$$

Now, $\vec{F}_{ii} = 0$ (self force)

So, total work done;

$$W = \sum_i W_i = \sum_i \sum_k \vec{F}_{ik} \cdot d\vec{r}_i \quad -(2)$$

Interchanging the order:

$$W = \sum_k \sum_i \vec{F}_{ki} \cdot d\vec{r}_k$$

So, $$W = \frac{1}{2} \sum_i \sum_k (\vec{F}_{ik} \cdot d\vec{r}_i + \vec{F}_{ki}\, d\vec{r}_k)$$

Newton's 3rd law: $\vec{F}_{ik} = -\vec{F}_{ki}$

So, $$W = \frac{1}{2} \sum_i \sum_k \vec{F}_{ik} (d\vec{r}_i - d\vec{r}_k)$$

~~$\Rightarrow W = \frac{1}{2} \sum_i \sum_k \vec{F}_{ik} \cdot d\vec{r}_{ik}$~~ $-(3)$ ~~$d\vec{r}_i - d\vec{r}_k = d\vec{r}_{ik}$~~

for rigid body; $|\vec{r_i} - \vec{r_k}| = $ const.

$\Rightarrow d\vec{r_i} - d\vec{r_k} = 0.$

so, $\boxed{W = 0.}$ for rigid body.

so, forces of constant in a rigid body don't do any work.

D'Alembert's Principle:

Total force on ith particle is;

$$\vec{F_i} = \underbrace{\vec{F_i}^a}_{\text{Applied force on ith particle}} + \vec{F_i}^c \rightarrow \text{Force of constant}$$

From NLM: $\vec{F_i} = \vec{F_i}^a + \vec{F_i}^c = \frac{d}{dt}\vec{P_i} = \dot{\vec{P_i}}$

Work done in a virtual displacement,

$W = \sum_i \vec{F_i}\delta\vec{r_i} = \sum_i \vec{F_i}^a . \delta\vec{r_i} + \sum_i \vec{F_i}^c . \delta\vec{r_i}$

$= \sum_i \dot{\vec{P_i}} . \delta\vec{r_i}$

Virtual w.d. by forces of const., $\sum_i \vec{F_i}^c \delta\vec{r_i} = 0.$

so, $\sum_i \vec{F_i}^a \delta\vec{r_i} - \sum_i \dot{\vec{P_i}} . \delta\vec{r_i} = 0.$

$\Rightarrow \sum_i (\vec{F_i}^a - \dot{\vec{P_i}}) . \delta\vec{r_i} = 0.$

$-\dot{\vec{P_i}}$, appears as an effective force, called the reverse of Inertia.

D'Alembert's Principle: $\underline{\sum_i (\vec{F_i}^a - \dot{\vec{P_i}}) . \delta\vec{r_i} = 0.}$

No. of eqn reduced as compared to NLM.

Potential en. (V) in an extremum at equim: conservative force; $\vec{F_i}^a = -\vec{\nabla} V.$

$V = V(\vec{r_1}, \vec{r_2}, \dots, \vec{r_n})$ (PE).

For conservative force,

$\sum_i \vec{F_i}^a . \delta\vec{r_i} = 0 \Rightarrow \sum (\vec{\nabla} V) . \delta\vec{r_i} = 0.$

so, $\delta V = 0$

δV is the change in PE for the virtual displacement.

$\delta V = 0 \Rightarrow V =$ minima: stable equim

$V =$ maxima: unstable equim.

$V \Rightarrow$ independent of $\vec{r_i}$: Neutral

1. Degrees of freedom (f): $f = 3N+k$

q_i : $i \Rightarrow 1$ to f.

Consider: $x_i = x_i(q_1, q_2, q_3, \ldots q_n, t)$. (independent quantities)

$x_{3N} = x_{3N}(q_1, q_2, q_3, \ldots, q_n, t)$

In terms of generalised position vector,

$\vec{r}_i = \vec{r}_i(q_1, q_2, q_3, \ldots, q_n, t)$

i: no. of particles, N: total no. of particles

(n: no. of dof.).

- The choice of the set of generalised co-ordinates is in no way unique, the different sets of choice will be functionally related;

$$q_i = q_i(q_1', q_2', q_3', \ldots, q_n').$$

$$d q_i = \sum_{k=1}^{n} \frac{\partial q_i}{\partial q_k'} dq_k'$$

Here q and q' represents two diff. sets of co-ordinates.

Jacobian; $J = \left(\frac{\partial(q_1, q_2, \ldots, q_n)}{\partial(q_1', q_2', \ldots, q_n')}\right) = \begin{vmatrix} \frac{\partial q_1}{\partial q_1'} & \cdots & \frac{\partial q_r}{\partial q_1'} & \cdots & \frac{\partial q_n}{\partial q_1'} \\ \vdots & & \vdots & & \vdots \\ \frac{\partial q_1}{\partial q_r'} & \cdots & \frac{\partial q_r}{\partial q_r'} & \cdots & \frac{\partial q_n}{\partial q_r'} \\ \vdots & & \vdots & & \vdots \\ \frac{\partial q_1}{\partial q_n'} & \cdots & \frac{\partial q_r}{\partial q_n'} & \cdots & \frac{\partial q_n}{\partial q_n'} \end{vmatrix}$

Lagrange's eqn of motion:

Let holonomic dynamical system consists of N no. of particles and n no. of dof.

$$\vec{r}_i = \vec{r}_i(q_1, q_2, \ldots q_n, t) \quad \text{--- (1)}.$$

The vel. of the ith particle is;

$$\vec{v}_i = \frac{d\vec{r}_i}{dt} = \sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j}\left(\frac{dq_j}{dt}\right) + \left(\frac{\partial r_i}{\partial t}\right). \quad \text{--- (2)}$$

$$= \frac{\partial r_i}{\partial q_1} \dot{q}_1 + \partial r_i/\partial q_2 \, \dot{q}_2 + \ldots + \partial r_i/\partial t.$$

Also, $\frac{\partial}{\partial \dot{q}_j}\left(\frac{d\vec{r_i}}{dt}\right) = \frac{\partial r_i}{\partial q_j}$ or, $\frac{\partial \vec{v_i}}{\partial \dot{q}_j} = \frac{\partial \vec{r_i}}{\partial q_j}$ — (3)

So, $\frac{\partial \vec{v_i}}{\partial \dot{q}_j} = \frac{\partial \vec{r_i}}{\partial q_j}$ | $\frac{\partial}{\partial \dot{q}_i}\frac{\partial r_i}{\partial t} = 0$ as, r_i is holonomic

Virtual displacement: $\delta\vec{r_i} = \sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j} \cdot \delta q_j$

Virtual work; $\delta W = \sum_{i=1}^{N} \vec{F_i} \cdot \delta\vec{r_i}$

$\Rightarrow \delta W = \sum_{j=1}^{n} \left(\sum_{i=1}^{N} \vec{F_i} \cdot \frac{\partial r_i}{\partial q_j} \right) \delta q_j$

$\Rightarrow \delta W = \sum_j Q_j \, \delta q_j$ — (4).

Generalised force; $Q_j = \sum_{i=1}^{N} F_i \cdot \frac{\partial r_i}{\partial q_j}$ — (5).

so; $\sum_i \vec{F_i} \cdot \delta\vec{r_i} = \sum_i m_i \ddot{\vec{r_i}} \cdot \delta\vec{r_i}$

$= \sum_j \left(\sum_i m_i \ddot{\vec{r_i}} \cdot \frac{\partial \vec{r_i}}{\partial q_j} \right) \delta q_j$

$\sum_i m_i \ddot{\vec{r_i}} \cdot \frac{\partial r_i}{\partial q_j} = \sum_i \left[\frac{d}{dt}\left(m_i \dot{\vec{r_i}} \cdot \frac{\partial \vec{r_i}}{\partial q_j}\right) - m_i \dot{\vec{r_i}} \frac{d}{dt}\left(\frac{\partial \vec{r_i}}{\partial q_j}\right) \right]$.

$\Rightarrow \frac{d}{dt}\left(\frac{\partial \vec{r_i}}{\partial q_j}\right) = \sum_i \frac{\partial^2 \vec{r_i}}{\partial q_i \partial q_j} \dot{q}_i + \frac{\partial}{\partial t}\left(\frac{\partial \vec{r_i}}{\partial q_j}\right)$.

$= \frac{\partial}{\partial q_j} \left[\sum_i \left(\frac{\partial r_i}{\partial q_i}\right) \dot{q}_i + \frac{\partial \vec{r_i}}{\partial t} \right]$. — (6).

Also; $\frac{\partial v_i}{\partial \dot{q}_j} = \frac{\partial \vec{r_i}}{\partial q_j}$

$\sum_i m_i \ddot{\vec{r_i}} \cdot \left(\frac{\partial \vec{r_i}}{\partial q_j}\right) = \sum \left\{ \frac{d}{dt}\left(m_i \vec{v_i} \cdot \frac{\partial \vec{v_i}}{\partial \dot{q}_j}\right) - m_i \vec{v_i} \cdot \left(\frac{\partial \vec{v_i}}{\partial q_j}\right) \right\}$

$= \frac{d}{dt}\left\{ \frac{\partial}{\partial \dot{q}_j}\left(\sum_i \frac{1}{2} m_i v_i^2 \right) \right\} - \frac{\partial}{\partial q_j}\left(\sum_i \frac{1}{2} m_i v_i^2 \right)$.

KE of sys. $T = \frac{1}{2} \sum m_i v_i^2$.

So, $\sum m_i \ddot{\vec{r_i}} \cdot \left(\frac{\partial \vec{r_i}}{\partial q_j}\right) = \frac{d}{dt}\left\{ \frac{\partial}{\partial \dot{q}_j} T \right\} - \frac{\partial T}{\partial q_j}$ — (7).

By using (5), ( ) and (7);

D'Alembert's principle becomes;

$$\sum_j Q_j \cdot \delta q_j - \sum_j \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right] \delta q_j = 0$$

$$\rightarrow \sum_j \left[Q_j - \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j}\right)\right] \delta q_j = 0. \quad \delta q_j \rightarrow (\neq 0).$$

So,

$$\boxed{\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j} \quad —(8)$$

$j : 1$ to n.

These are the Lagrange's eq^n of motion of the 2nd kind. This is valid for both conservative and dissipative or non-conservative forces.

- For the conservative forces;

$$V = V(r_1, r_2, r_3, \ldots r_N).$$
$$= V(q_1, q_2, q_3, \ldots, q_n).$$

So, $dV = \sum_i \left[\left(\frac{\partial V}{\partial x_i}\right) dx_i + \left(\frac{\partial V}{\partial y_i}\right) dy_i + \left(\frac{\partial V}{\partial z_i}\right) dz_i\right].$

$$\rightarrow \text{so, } \frac{\partial V}{\partial q_j} = \sum_i \vec{\nabla}_i V \cdot \frac{\partial r_i}{\partial q_j}$$

Gen. force; $Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = -\sum_i (\vec{\nabla}_i V) \cdot \frac{\partial \vec{r}_i}{\partial q_j}$

$$\rightarrow Q_j = -\frac{\partial V}{\partial q_j}$$

$$(8) \rightarrow \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j = -\frac{\partial V}{\partial q_j}$$

$$\rightarrow \frac{d}{dt}\frac{\partial (T-V)}{\partial \dot{q}_j} - \frac{\partial}{\partial q_j}(T-V) = 0 \quad \Big| \quad \frac{\partial V}{\partial \dot{q}_j} = 0.$$

$$\rightarrow \boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0} \quad —(9) \quad \begin{matrix}(n < 3N) \\ (j : 1 \text{ to } n) \\ (n = 3N - k)\end{matrix}$$

where, $L = T - V$ is the Lagrangian of the system.

It's the Lagrange's eq^n of motion for conservative forces.

Newtonian: $3N$ eq^n.

Lagrangian: $(3N-k)$ eq^n. (Less no. of eq^n)

- L is not unique. Consider; $L'(q,\dot{q},t)$

$$L'(q,\dot{q},t) = L(q,\dot{q},t) + \frac{df}{dt} \quad -(10)$$

$\left(\frac{d}{dt}\frac{\partial L'}{\partial \dot{q}_j} - \frac{\partial L'}{\partial q_j} = 0\right)$ satisfies same eq^n of motion.

Ex. 1D Harmonic Oscillator:

(frictionless)

Force; $F = -kx$.

$V_{(x)} = -\int F\,dx = \frac{1}{2}kx^2$

KE; $T = \frac{1}{2}m\dot{x}^2$; $L = L(x,\dot{x},t)$.

$$L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

$$\frac{\partial L}{\partial x} = -kx, \quad \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

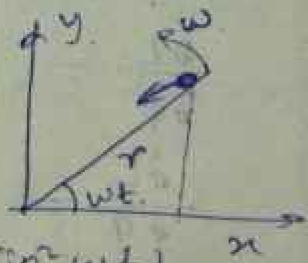
So $(9) \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$.

$$\Rightarrow \frac{d}{dt}m\dot{x} - (-kx) = 0.$$

$$\Rightarrow m\ddot{x} + kx = 0. \text{ or, } \ddot{x} = -\left(\frac{k}{m}\right)x.$$

(eq^n of SHM). or, $\ddot{x} = -\omega^2 x$.

Q. A bead is sliding along a smooth st. wire which is uniformly rotating about z-axis in a force free space.



$\rightarrow$ $x = r\cos\omega t$, $y = r\sin\omega t$.

$\dot{x} = \dot{r}\cos\omega t - r\omega\sin\omega t$
$\dot{y} = \dot{r}\sin\omega t + r\omega\cos\omega t$ | (ω = const.)

$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) = \frac{m\dot{r}^2}{2}(\cos^2\omega t + \sin^2\omega t) + \frac{mr^2\omega^2}{2}(\sin^2\omega t + \cos^2\omega t) + \frac{mr\dot{r}\omega}{2}(2\sin\omega t\cos\omega t - 2\sin\omega t\cos\omega t)$$

$$\Rightarrow T = \frac{m\dot{r}^2}{2} + \frac{mr^2\omega^2}{2}; \quad V = 0 \text{ for force free region.}$$

$$L = T - V = \frac{m\dot{r}^2}{2} + \frac{mr^2\omega^2}{2}.$$

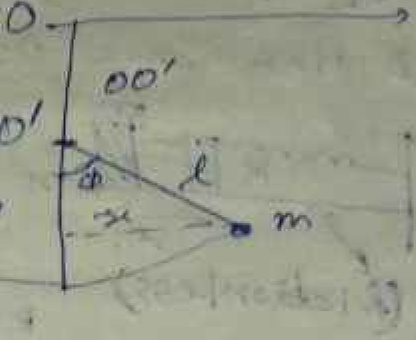
So, $\frac{\partial L}{\partial \dot{r}} = m\dot{r}$, $\frac{\partial L}{\partial r} = m\omega^2 r$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \Rightarrow m\ddot{r} - m\omega^2 r = 0$.

$\rightarrow \ddot{r} = \omega^2 r$ (eq^n of motion).

Q. A pendulum with a moving support:

Ans → Point of suspension O′ is attached to a moving lift which falls vertically with an accn 'f'.



$OO' = \frac{1}{2}ft^2$

Co-ordinate of COM $= (x, \frac{1}{2}ft^2 + l\cos\phi)$ (at any time t).

$x \downarrow l\sin\phi$; $\rightarrow -y$ actually.

$x = l\sin\phi$, $y = \frac{1}{2}ft^2 + l\cos\phi$

$\dot{x} = l\cos\phi\,\dot{\phi}$, $\dot{y} = ft - l\sin\phi\,\dot{\phi}$

$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{m}{2}(l^2\cos^2\phi\,\dot{\phi}^2 + l^2\sin^2\phi\,\dot{\phi}^2 + f^2t^2 - 2ftl\sin\phi\,\dot{\phi})$

$\Rightarrow T = \frac{m}{2}(l^2\dot{\phi}^2 - 2ftl\sin\phi\,\dot{\phi} + f^2t^2)$

$V = -mg(\frac{1}{2}ft^2 + l\cos\phi)$; $(-mgy)$.

$L = T - V = \frac{m}{2}(l^2\dot{\phi}^2 - 2ftl\sin\phi\,\dot{\phi} + f^2t^2) + mg(\frac{1}{2}ft^2 + l\cos\phi)$.

$\frac{\partial L}{\partial \dot{\phi}} = ml^2\dot{\phi} - mftl\sin\phi$

$\frac{\partial L}{\partial \phi} = -mftl\cos\phi\,\dot{\phi} - mgl\sin\phi$.

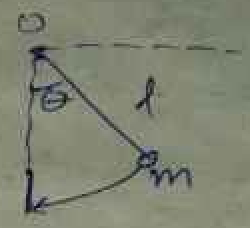
Now, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0 \Rightarrow (ml^2\ddot{\phi} - mlf\sin\phi - mlft\cos\phi\,\dot{\phi}) + mlft\cos\phi\,\dot{\phi} + mgl\sin\phi = 0$.

$\Rightarrow ml^2\ddot{\phi} - mlf\sin\phi + mgl\sin\phi = 0$.

$\Rightarrow ml^2\ddot{\phi} = ml\sin\phi\,(f-g)$.

$\Rightarrow \ddot{\phi} + \frac{g-f}{l}\sin\phi = 0.$ (Ans.)

Simple pendulum:



$x = l\sin\theta,\ y = l\cos\theta.$

$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) = \frac{ml^2}{2}\dot{\theta}^2$

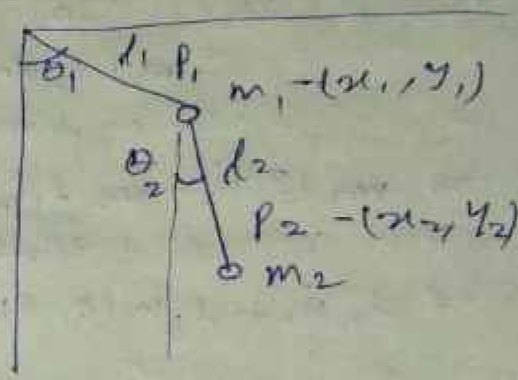
$V = -mgy = -mgl\cos\theta$

$L = T - V = \frac{ml^2\dot{\theta}^2}{2} + mgl\cos\theta.$

$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0 \Rightarrow \frac{d}{dt}(ml^2\dot{\theta}) + mgl\sin\theta = 0.$

$\Rightarrow ml^2\ddot{\theta} + mgl\sin\theta = 0 \Rightarrow \ddot{\theta} + \left(\frac{g}{l}\right)\sin\theta = 0.$ (Ans.)

② Double pendulum:



$x_1 = l_1\sin\theta_1$

$y_1 = l_1\cos\theta_1$

$x_2 = l_1\sin\theta_1 + l_2\sin\theta_2$

$y_2 = l_1\cos\theta_1 + l_2\cos\theta_2$

$T = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2).$

$\Rightarrow T = \frac{m_1}{2}l_1^2\dot{\theta}_1^2 + \frac{m_2}{2}\left[(l_1\cos\theta_1\dot{\theta}_1 + l_2\cos\theta_2\dot{\theta}_2)^2 + (l_1\sin\theta_1\dot{\theta}_1 - l_2\sin\theta_2\dot{\theta}_2)^2\right]$

$\Rightarrow T = \frac{m_1l_1^2}{2}\dot{\theta}_1^2 + \frac{m_2}{2}\left[l_1^2\cos^2\theta_1\dot{\theta}_1^2 + l_2^2\cos^2\theta_2\dot{\theta}_2^2 + 2l_1l_2\cos\theta_1\cos\theta_2\dot{\theta}_1\dot{\theta}_2 + l_1^2\sin^2\theta_1\dot{\theta}_1^2 + l_2^2\sin^2\theta_2\dot{\theta}_2^2 + 2l_1l_2\sin\theta_1\sin\theta_2\dot{\theta}_1\dot{\theta}_2\right].$

$\Rightarrow T = \frac{1}{2}m_1l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2l_2^2\dot{\theta}_2^2 + m_2l_1l_2\dot{\theta}_1\dot{\theta}_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)$

$\Rightarrow T = \frac{1}{2}(m_1+m_2)l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2l_2^2\dot{\theta}_2^2 + m_2l_1l_2\cos(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2$

Now, $V = -m_1gy_1 - m_2gy_2 = -m_1gl_1\cos\theta_1 - m_2g(l_1\cos\theta_1 + l_2\cos\theta_2)$

$\Rightarrow V = -(m_1+m_2)gl_1\cos\theta_1 - m_2gl_2\cos\theta_2.$

$L = T - V = \frac{1}{2}(m_1+m_2)l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2l_2^2\dot{\theta}_2^2 + m_2l_1l_2\cos(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2 + (m_1+m_2)gl_1\cos\theta_1 + m_2gl_2\cos\theta_2.$

So; $\frac{\partial L}{\partial \dot{\theta}_1} = (m_1+m_2) l_1^2 \dot{\theta}_1 + m_2 l_1 l_2 \cos(\theta_1-\theta_2) \dot{\theta}_2$

$\frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2 \dot{\theta}_2 + m_2 l_1 l_2 \cos(\theta_1-\theta_2) \dot{\theta}_1$

$\frac{\partial L}{\partial \theta_1} = - m_2 l_1 l_2 \sin(\theta_1-\theta_2) \dot{\theta}_1 \dot{\theta}_2 - (m_1+m_2) g l_1 \sin\theta_1$

$\frac{\partial L}{\partial \theta_2} = m_2 l_1 l_2 \sin(\theta_1-\theta_2) \dot{\theta}_1 \dot{\theta}_2 - m_2 g l_2 \sin\theta_2$

$\underline{\theta_1}:$ $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = 0 \Rightarrow (m_1+m_2) l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \cos(\theta_1-\theta_2) \ddot{\theta}_2$

$- m_2 l_1 l_2 \sin(\theta_1-\theta_2) \dot{\theta}_2 (\dot{\theta}_1 - \dot{\theta}_2) + m_2 l_1 l_2 \sin(\theta_1-\theta_2) \dot{\theta}_1 \dot{\theta}_2$

$+ (m_1+m_2) g l_1 \sin\theta_1 = 0$

$\Rightarrow (m_1+m_2) l_1 \ddot{\theta}_1 + m_2 l_2 \cos(\theta_1-\theta_2) \ddot{\theta}_2 + m_2 l_2 \sin(\theta_1-\theta_2) \dot{\theta}_2^2$

$+ (m_1+m_2) g \sin\theta_1 = 0$ —— (1)

$\underline{\theta_2}:$ $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} = 0 \Rightarrow m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \cos(\theta_1-\theta_2) \ddot{\theta}_1$

$- m_2 l_1 l_2 \sin(\theta_1-\theta_2) \dot{\theta}_1 (\dot{\theta}_1 - \dot{\theta}_2) - m_2 l_1 l_2 \sin(\theta_1-\theta_2) \dot{\theta}_1 \dot{\theta}_2$

$+ m_2 g l_2 \sin\theta_2 = 0$

$\Rightarrow \cancel{m_2} l_2^2 \ddot{\theta}_2 + \cancel{m_2} l_1 l_2 \cos(\theta_1-\theta_2) \ddot{\theta}_1 - \cancel{m_2} l_1 l_2 \sin(\theta_1-\theta_2) \dot{\theta}_1^2$

$+ g l_2 \sin\theta_2 = 0$

$\Rightarrow l_2 \ddot{\theta}_2 + l_1 \cos(\theta_1-\theta_2) \ddot{\theta}_1 - l_1 \sin(\theta_1-\theta_2) \dot{\theta}_1^2 + g \sin\theta_2 = 0$

—— (2)

(1) $\rightarrow l_1 \ddot{\theta}_1 + g \sin\theta_1 + \left(\frac{m_2}{m_1+m_2}\right) l_2 [\cos(\theta_1-\theta_2) \ddot{\theta}_2 + \sin(\theta_1-\theta_2) \dot{\theta}_2^2] = 0$

(2) $\rightarrow l_2 \ddot{\theta}_2 + g \sin\theta_2 + l_1 [\cos(\theta_1-\theta_2) \ddot{\theta}_1 - \sin(\theta_1-\theta_2) \dot{\theta}_1^2] = 0$

Special case: $m_1 = m_2 = m$, $l_1 = l_2 = l$;

(1) $\rightarrow 2(\ddot{\theta}_1 + \frac{g}{l} \sin\theta_1) + \cos(\theta_1-\theta_2) \ddot{\theta}_2 + \sin(\theta_1-\theta_2) \dot{\theta}_2^2 = 0$

(2) $\rightarrow \ddot{\theta}_2 + \frac{g}{l} \sin\theta_2 + \cos(\theta_1-\theta_2) \ddot{\theta}_1 - \sin(\theta_1-\theta_2) \dot{\theta}_1^2 = 0$

For small θ; $\cos(\theta_1-\theta_2) \simeq 1$, $\sin(\theta_1-\theta_2) \simeq \theta_1 - \theta_2$

So; (1) $\rightarrow 2l\ddot{\theta}_1 + 2g\theta_1 + l\ddot{\theta}_2 + l\theta_1\dot{\theta}_2^2 - l\theta_2\dot{\theta}_2^2 = 0$

(2) $\rightarrow l\ddot{\theta}_2 + g\theta_2 + l\ddot{\theta}_1 - l\theta_1\dot{\theta}_1^2 + l\theta_2\dot{\theta}_1^2 = 0$

$(\theta_1-\theta_2)\dot{\theta}_1^2$ and $(\theta_1-\theta_2)\dot{\theta}_2^2$ can be neglected.

So; $\cancel{2l\ddot{\theta}_1 + l\theta}$ $2\ddot{\theta}_1 + \ddot{\theta}_2 = -2\frac{g}{l}\theta_1$

and $\ddot{\theta}_1 + \ddot{\theta}_2 = -\frac{g}{l}\theta_2$

Spherical polar co-ordinate (r, θ, ϕ):

$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta.$

$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2)$

$\cancel{V = mgz = mgr\cos\theta}$. Let, $V = \frac{1}{2}kr^2$.

$L = T - V = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2) - \frac{1}{2}kr^2.$

$\frac{\partial L}{\partial \dot{r}} = m\dot{r},\ \frac{\partial L}{\partial r} = mr\dot{\theta}^2 + mr\sin^2\theta\,\dot{\phi}^2 - kr.$

$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta},\ \frac{\partial L}{\partial \theta} = mr^2\sin\theta\cos\theta\,\dot{\phi}^2$

$\frac{\partial L}{\partial \dot{\phi}} = mr^2\sin^2\theta\,\dot{\phi},\ \frac{\partial L}{\partial \phi} = 0$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \Rightarrow m\ddot{r} - mr\dot{\theta}^2 - mr\sin^2\theta\,\dot{\phi}^2 + kr = 0.$

$\Rightarrow \ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\,\dot{\phi}^2 + \frac{k}{m}r = 0 \quad \text{—(1)}.$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \Rightarrow mr^2\ddot{\theta} + 2mr\dot{\theta}\dot{r} - mr^2\sin\theta\cos\theta\,\dot{\phi}^2 = 0.$

$\Rightarrow r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} - r^2\sin\theta\cos\theta\,\dot{\phi}^2 = 0. \quad \text{—(2)}.$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0 \Rightarrow mr^2\sin^2\theta\,\ddot{\phi} + 2mr\sin^2\theta\,\dot{r}\dot{\phi} + mr^2\sin 2\theta\,\dot{\theta}\dot{\phi} = 0. \quad \text{—(3)}$

• Compound pendulum: (see from book)
(A B Gupta - P 258).

Q. A bead is sliding on a wire in the shape of a cycloid described by eq^n; ~~x=~~

$x = a(\theta - \sin\theta),\ y = a(1 + \cos\theta)$

where θ ~~...~~ runs from 0 to 2π.

$\rightarrow$ $x = a(\theta - \sin\theta)$

$y = a(1 + \cos\theta).$

$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) = \frac{ma^2}{2}(\dot{\theta}^2 + \cos^2\theta\,\dot{\theta}^2 - 2\dot{\theta}\cos\theta\,\dot{\theta} + \sin^2\theta\,\dot{\theta}^2)$

$= \frac{ma^2\dot{\theta}^2}{2}(1 - 2\cos\theta + 1)$

$\Rightarrow T = ma^2\dot{\theta}^2(1 - \cos\theta).$

$V = mgy = mga(1+\cos\theta)$

$L = ma^2\dot\theta^2(1-\cos\theta) - mga(1+\cos\theta)$

$\frac{\partial L}{\partial\dot\theta} = 2ma^2\dot\theta(1-\cos\theta)$

$\frac{\partial L}{\partial\theta} = ma^2\dot\theta^2\sin\theta + mga\sin\theta$

$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) - \frac{\partial L}{\partial\theta} = 0 \Rightarrow 2ma^2\ddot\theta(1-\cos\theta) + 2ma^2\sin\theta\,\dot\theta^2 - ma^2\sin\theta\,\dot\theta^2 - mga\sin\theta = 0.$

$\Rightarrow 2ma^2\ddot\theta(1-\cos\theta) + ma^2\sin\theta\,\dot\theta^2 - mga\sin\theta = 0$

$\Rightarrow 2\ddot\theta(1-\cos\theta) + \sin\theta\,\dot\theta^2 - \frac{g}{a}\sin\theta = 0.$

$\Rightarrow \ddot\theta(1-\cos\theta) + \frac{1}{2}\sin\theta\,\dot\theta^2 - \frac{g}{2a}\sin\theta = 0.$

2. L is not unique: $L' = L + \frac{dF}{dt}(q,\dot q,t)$

The difference in their variation is

$\delta L' - \delta L = \delta\frac{dF}{dt}(q,\dot q,t)$

~~which as the variation~~

The variation of the action is given by,

$\delta S = \int \delta L\,dt$

Diff: $\delta S' - \delta S = \int(\delta L' - \delta L)\,dt$

$\Rightarrow$ i.e. $= \int_{t_1}^{t_2}\frac{d}{dt}\delta F(q,\dot q,t)\,dt = \delta F(q,\dot q,t)\Big|_{t_1}^{t_2}$

Nature of variation problem is such that the variation vanishes at the end points.

$\delta S' - \delta S = \delta F(q,\dot q,t)\Big|_{t_1}^{t_2} = \delta F(q,\dot q,t_2) - \delta F(q,\dot q,t_1)$

$\Rightarrow$ so; $\delta F(q,\dot q,t_2) = \delta F(q,\dot q,t_1) = 0$

so, $\delta S' - \delta S = 0$

The Euler Lagrangian eqn corresponding to L & L' will be same; i.e.

$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot q}\right) - \frac{\partial L}{\partial q} = 0 \Rightarrow \frac{d}{dt}\left(\frac{\partial L'}{\partial\dot q}\right) - \frac{\partial L'}{\partial q} = 0.$

Generalised Momentum:

$\frac{\partial L}{\partial\dot x_i} = \frac{\partial T}{\partial\dot x_i} - \frac{\partial V}{\partial\dot x_i}$; $[V = V(x_i)]$

$= \frac{\partial}{\partial\dot x_i}\sum_{i=1}^{n}\frac{1}{2}m_i\dot x_i^2$

$\Rightarrow \frac{\partial L}{\partial \dot{x}_i} = m_i \dot{x}_i = p_{x_i}$

Generalised momentum (p_j) associated with generalised co-ordinate (q_j) is defined as;

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \quad ; \; j : 1 \text{ to } n.$$

If Lagrangian of a sys. doesn't contain a given co-ordinate (q_j) although it may contain corresponding velocity ($\dot{q}_j$). Then, the co-ordinate is said to be cyclic or ignorable co-ordinate.

- For a cyclic co-ordinate Lagrange eqn reduces to $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0$; as $\frac{\partial L}{\partial q_j} = 0$.

$\Rightarrow \frac{d}{dt} p_j = 0 \Rightarrow p_j = \text{const.}$
(mom. is conserved)

central force: $L = T - V = \frac{1}{2}(m\dot{r}^2 + mr^2\dot{\theta}^2) - V(r)$

(L depends on r; not θ)

$\frac{d}{dt} p_\theta = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{d}{dt} mr^2\dot{\theta} = 0. \Rightarrow mr^2\dot{\theta} = p_\theta = \text{const.}$
(corresponding mom.). (Here p_θ is conserved).
(Ang. mom)

Particle in an EM field:

The Lagrangian is; $L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - e\phi(x,y,z,t) + e\vec{A}(x,y,z,t).\vec{v}$

$\phi \Rightarrow$ electric scalar pot.
$\vec{A} \Rightarrow$ magnetic vector pot.

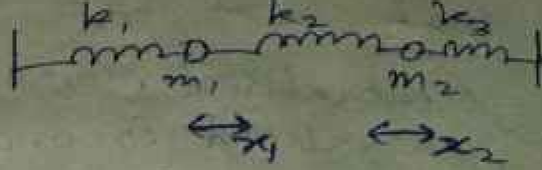
$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}, \cancel{p_y = m\dot{y}} + eA_x$

$p_y = m\dot{y} + eA_y, \quad p_z = m\dot{z} + eA_z$

mechanical mom.

Additional ~~field~~ mom.

Coupled Masses:



$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2$$

$$V = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2x_2^2 + \frac{1}{2}k_3(x_2-x_1)^2$$

So, $L = T - V = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - \frac{1}{2}k_1x_1^2 - \frac{1}{2}k_2x_2^2$
$- \frac{1}{2}k_3(x_2-x_1)^2$

$\frac{\partial L}{\partial \dot{x}_1} = m_1\dot{x}_1$, $\frac{\partial L}{\partial \dot{x}_2} = m_2\dot{x}_2$

$\frac{\partial L}{\partial x_1} = -k_1x_1 + k_3(x_2-x_1)$, $\frac{\partial L}{\partial x_2} = -k_2x_2 - k_3(x_2-x_1)$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0 \Rightarrow m_1\ddot{x}_1 + k_1x_1 - k_3(x_2-x_1) = 0$

$\Rightarrow m_1\ddot{x}_1 + (k_1+k_3)x_1 - k_3x_2 = 0$ — (1)

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = 0 \Rightarrow m_2\ddot{x}_2 + k_2x_2 + k_3x_2 - k_3x_1 = 0$

$\Rightarrow m_2\ddot{x}_2 + (k_2+k_3)x_2 - k_3x_1 = 0$. — (2).

Hamiltonian Formulation

Hamiltonian eq^{n}s:
Lagrange formulation is in terms of gen. ~~co-ord~~ co-ordinates (q_s) and gen. vel. ($\dot{q}_s$).
The eq^n of motion are like Newton's 2nd law. ~~Hamil~~ (2nd order in time); n eq^{n}s $\boxed{s: 1 \text{ to } n}$
Hamiltonian formulation is in terms of gen. co-ordinates (q_s) and gen. mom. (p_s), as independent variable and we would get, $2n$ 1st order ~~eqn~~ eq^n.

$$L = L(q_s, \dot{q}_s, t) \; ; \; \frac{dL}{dt} = \sum_{s=1}^{n}\left[\frac{\partial L}{\partial q_s}\cdot\frac{dq_s}{dt} + \frac{\partial L}{\partial \dot{q}_s}\frac{d\dot{q}_s}{dt}\right] + \frac{\partial L}{\partial t}$$

$$\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} = 0\right].$$

So, $$\frac{dL}{dt} = \sum_{s=1}^{n}\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right)\dot{q}_s + \frac{\partial L}{\partial \dot{q}_s}\frac{d\dot{q}_s}{dt}\right] + \frac{\partial L}{\partial t}$$

$$\Rightarrow \frac{dL}{dt} = \sum_{j=1}^{n}\left[\frac{d}{dt}\left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right) - \cancel{\frac{\partial L}{\partial q_j}\dot{q}_j + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) q_j} + \frac{\partial L}{\partial t}\right]$$

$$\Rightarrow \frac{d}{dt}\left[\sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L\right] + \frac{\partial L}{\partial t} = 0$$

If $\underline{\frac{\partial L}{\partial t} = 0}$; ($L$ is explicitly independent of time).

$$\sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L = \text{const.} = H.$$

So, Hamiltonian: $\boxed{H = \sum_{j=1}^{n} \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L}$

If transformation eqn donot contain any time explicitly, ~~T is always a~~ the constraints are time independent. T is always a homogeneous quadratic fn of gen. vel.

$\sum x_i \frac{\partial f}{\partial x_i} = nf \Rightarrow$ Euler's theorem.

f: Homogeneous fn of degree n

So, $\sum_j \dot{q}_j \left(\frac{\partial L}{\partial \dot{q}_j}\right) = \sum_j \dot{q}_j \left(\frac{\partial T}{\partial \dot{q}_j}\right) = 2T$

[$f: T$; $x_i: \dot{q}_j$; $n = 2$ as $T = \frac{1}{2}m\dot{x}^2$]

So, $H = 2T - L = 2T - (T - V) \Rightarrow \boxed{H = T + V}$

So, Hamiltonian is the total mechanical en. of the system.

Hamiltonian Eqn of Motion:

$$H = \sum_j \dot{q}_j \left(\frac{\partial L}{\partial \dot{q}_j}\right) - L = \sum_j p_j \dot{q}_j - L \quad \text{—(1)}$$

$$\Rightarrow dH = \sum_j \dot{q}_j dp_j + \sum_j p_j d\dot{q}_j - \left[\sum_j \left(\frac{\partial L}{\partial q_j}\right) dq_j + \sum_j \left(\frac{\partial L}{\partial \dot{q}_j}\right) d\dot{q}_j + \frac{\partial L}{\partial t} dt\right]$$

$H(q_j, p_j, t)$

$$= \sum_j \dot{q}_j dp_j + \sum_j p_j d\dot{q}_j - \sum_j \left(\frac{\partial L}{\partial q_j}\right) dq_j - \sum_j p_j d\dot{q}_j - \frac{\partial L}{\partial t} dt$$

$$\Rightarrow \underline{dH = \sum \dot{q}_j dp_j - \sum \dot{p}_j dq_j - \frac{\partial L}{\partial t} dt} \quad \text{—(2)}$$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)$

$\sum \frac{\partial L}{\partial q_j} = \frac{d}{dt} \sum \frac{\partial L}{\partial \dot{q}_j}$

Now; $H = H(q_j, p_j, t)$.

$\Rightarrow dH = \sum \frac{\partial H}{\partial p_j} dp_j + \sum_j \frac{\partial H}{\partial q_j} dq_j + \frac{\partial H}{\partial t} dt$ —(3)

Comparing (2) and (3);

$\boxed{\dot{q}_j = \frac{\partial H}{\partial p_j}}$, $\boxed{\dot{p}_j = -\frac{\partial H}{\partial q_j}}$, $\boxed{\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}}$

$(j: 1 \text{ to } n)$ d.o.f $\quad$ (n) eqⁿ $\quad$ (n) eqⁿ $\quad$ 1 eqⁿ

No. of eqⁿ $= n + n + 1 = (2n+1)$.

These are Hamiltonian canonical eqⁿ or Hamiltonian eqⁿ of motion.

(2n+1) 1st order Diff. Eqⁿ.

Lagrangian: n no. of 2nd order DE; 2n initial conditions required.

Also, $\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \Rightarrow \frac{\partial}{\partial t}(H+L) = 0 \Rightarrow H+L = \text{const}$.

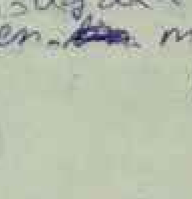
So; H+L is explicitly independent of time.

In Hamiltonian formulation, there are 2n 1st order DE in 2n independent variables, n gen. co-ordinate (q_j) and n gen. conjugate mom. (p_j).

$p_j = \partial L / \partial \dot{q}_j$

Applications:

Compound Pendulum: (step to step...)



#1 $L = \frac{1}{2} I \dot{\theta}^2 - (-mgL\cos\theta)$

#2 $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = I\dot{\theta} \Rightarrow \dot{\theta} = \frac{p_\theta}{I}$

So, #3 $H = \sum p_j \dot{q}_j - L = p_\theta \dot{\theta} - \frac{1}{2} I \dot{\theta}^2 - mgL\cos\theta$

$\Rightarrow H = \frac{p_\theta^2}{I} - \frac{p_\theta^2}{2I} - mgL\cos\theta = \frac{p_\theta^2}{2I} - mgL\cos\theta$

$\Rightarrow H = \frac{p_\theta^2}{2I} - mgL\cos\theta$ (time independent)

So, Hamiltonian eqⁿs of motion are;

#4 $\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{I}$ $\Rightarrow p_\theta = I\dot{\theta}$

$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -mgL\sin\theta$.

$\Rightarrow I\ddot{\theta} = -mgL\sin\theta$

$\Rightarrow \ddot{\theta} + \frac{mgL}{I}\sin\theta = 0$.

eqⁿ of motion.

Equivalence of Newton's formulation with Lagrange's & Hamilton's:

(a) $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0$; $x_i = \vec{x}_1, \vec{x}_2, \vec{x}_3$ (x, y, z)

$\Rightarrow \frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}_i}\right)(T-V) - \frac{\partial}{\partial x_i}(T-V) = 0$ | $\frac{\partial V}{\partial \dot{x}_i} = 0$

$\Rightarrow \frac{d}{dt}\frac{\partial T}{\partial \dot{x}_i} + \frac{\partial V}{\partial x_i} = 0$ | $\frac{\partial T}{\partial x_i} = 0$

$T = \frac{1}{2} m_i \dot{x}_i^2 \Rightarrow \frac{\partial L}{\partial \dot{x}_i} = m_i \dot{x}_i = p_i$

$V(x_i) \Rightarrow$ conservative sys

$\Rightarrow F_i = -\partial V/\partial x_i$

So, $\frac{d}{dt} p_i - F = 0 \Rightarrow \underline{F = \frac{dp_i}{dt}}$ | Newton's 2nd Law of motion

(b) ~~$\dot{q}_j = \frac{\partial H}{\partial p_j} \Rightarrow \dot{x}_j = \frac{\partial T}{\partial p_j}$~~

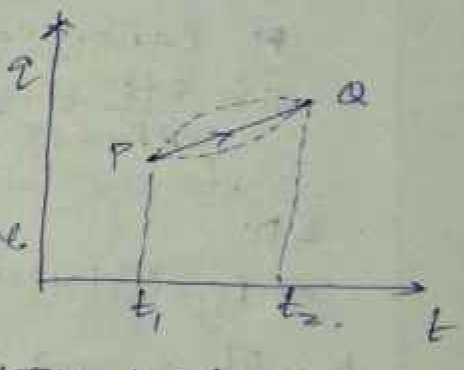
$\dot{p}_j = -\frac{\partial H}{\partial q_j} \Rightarrow \dot{p}_j = -\frac{\partial}{\partial x_j}(T+V) = -\frac{\partial V}{\partial x_j} = F_j$; conservative sys

So, $F_j = \frac{dp_j}{dt} \Rightarrow$ (NLM-2)

Principle of Least Action:

Consider a conservative system that moves from initial point P to final point Q.

The actual path of motion is along the solid path (line). The other curves show the varying neighbouring paths with the same ~~exp~~ end points P and Q.



~~Integ~~ Integral: $\underline{A = \int_{t_1}^{t_2} 2T\,dt}$; ($T \Rightarrow KE$) is called Action.

Conservative sys.: $H + L = T + V + T - V = 2T$

Also; $H + L = \sum_j \left(\frac{\partial L}{\partial \dot{q}_j}\right) d\dot{q}_j = \sum_j p_j \dot{q}_j$

$\Rightarrow 2T = \sum_i p_i \dot{q}_i$

So, $A = \int_{t_1}^{t_2} \sum_i p_i \dot{q}_i \, dt$

The principle of least action says that the variation along the actual path b/w the given time interval is minimum.

So, $\Delta A = \Delta \int_{t_1}^{t_2} \sum_i p_i \dot{q}_i \, dt = 0$.

This is mathematical statement of least action for a conservative sys.

Also, $A = \int_{t_1}^{t_2} (H+L)\, dt = \int_{t_1}^{t_2} L\,dt + \int_{t_1}^{t_2} H\,dt$

For holonomic sys; $\sum_i p_i \dot{q}_i = 2T$

$\Delta \int_{t_1}^{t_2} 2T\,dt = 0 \Rightarrow 2T\,[\Delta(t_2 - t_1)] = 0$

So, $(t_2 - t_1)$ is extremum (time)

Application of Hamiltonian:

Harmonic Osc. in 1D:

$T = \frac{1}{2} m\dot{x}^2$, $V = \frac{1}{2} m\omega^2 x^2$, $L = T - V = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} m\omega^2 x^2$

$\frac{\partial L}{\partial \dot{x}} = m\dot{x} = p_x \Rightarrow \dot{x} = \frac{p_x}{m}$

So, $L = \frac{p_x^2}{2m} - \frac{1}{2} m\omega^2 x^2$

$H = \left(\frac{\partial L}{\partial \dot{x}}\right)\dot{x} - L = p_x \dot{x} - L = p_x \cdot \frac{p_x}{m} - \frac{p_x^2}{2m} + \frac{1}{2} m\omega^2 x^2$

$\Rightarrow H = \frac{p_x^2}{2m} + \frac{1}{2} m\omega^2 x^2$ or, $H = T + V$

canonical eq^n:

$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m}$, $\dot{p}_x = -\frac{\partial H}{\partial x} = -m\omega^2 x$

$\Rightarrow \dot{p}_x + m\omega^2 x = 0$

$\Rightarrow m\ddot{x} + m\omega^2 x = 0$ - (eq^n)

2D:

$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}(k_1 x^2 + k_2 y^2)$

$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$, $p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y}$

So $H = \sum \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L = p_x \dot{x} + p_y \dot{y} - \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}(k_1 x^2 + k_2 y^2)$

$\Rightarrow H = \frac{p_x^2}{m} + \frac{p_y^2}{m} - \frac{p_x^2}{2m} - \frac{p_y^2}{2m} + \frac{1}{2}(k_1 x^2 + k_2 y^2)$

$\Rightarrow H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2}(k_1 x^2 + k_2 y^2)$

Eq^n of motion:

$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m}$

$\dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y}{m}$

$\dot{p}_x = -\frac{\partial H}{\partial x} = -k_1 x$

$\dot{p}_y = -\frac{\partial H}{\partial y} = -k_2 y$

$\Rightarrow m\ddot{x} + k_1 x = 0$, $m\ddot{y} + k_2 y = 0$

Hamilton's Principle:

All possible path along which a dynamical sys. may move from one point to another in a given time interval (consistent with constraints, if any), the system actually follows the one that minimise the time interval of the Lagrangian.

$$\int_{t_1}^{t_2} L\,dt = J \quad --- (1)$$

is an extremum (i.e. stationary) for the correct path of the motion.

$\delta J = 0$. so, $\delta \int_{t_1}^{t_2} (T-V)\,dt = \delta J = 0$

J: Hamiltonian principle fn

It's also known as principle of least action.

Modified Hamiltonian's principle:

$$\delta \int_{t_1}^{t_2} L\,dt = 0 = \delta \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)\,dt = 0$$

Again: $H(q_i, p_i, t) = \sum p_i \dot{q}_i - L(q_i, \dot{q}_i, t)$

$\Rightarrow L(q_i, \dot{q}_i, t) = \sum p_i \dot{q}_i - H(q_i, p_i, t)$

so; $\delta \int_{t_1}^{t_2} (\sum_i p_i \dot{q}_i - H)\,dt = 0$ Modified Hamiltonian Principle.

Lagrange's eqn from Hamiltonian principle:

$L = L(q_i, \dot{q}_i, t)$

$$\Rightarrow \delta L = \sum_{i=1}^{n} \frac{\partial L}{\partial q_i}\delta q_i + \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i + \frac{\partial L}{\partial t}\delta t$$

~~If~~ For no time variation; $\frac{\partial L}{\partial t}\delta t = 0$
(virtual displacement)

so; $\int_{t_1}^{t_2} \delta L\,dt = \sum \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i\right) dt = 0.$

$$\Rightarrow \int_{t_1}^{t_2} \sum \frac{\partial L}{\partial q_i}\delta q_i\,dt + \int_{t_1}^{t_2} \sum \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i\,dt = 0$$ (by Hamilton's principle)

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i\,dt = \left[\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right]_{t_1}^{t_2} - \int \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\delta q_i\,dt.$$

(By parts)

$$\left[\int \delta \dot{q}_i\,dt = \int \frac{d}{dt}(\delta q_i)\,dt = \delta q_i\right].$$

But at the two ends of the path, q_i at t_1 and t_2 doesn't vary. $\delta q_i = 0$ at $t = t_1$ and $t = t_2$.

So; $\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial \dot{q}_i}\right)\delta \dot{q}_i \, dt = -\int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i \, dt$

So; $\int_{t_1}^{t_2} \sum \frac{\partial L}{\partial q_i}\delta q_i \, dt - \int_{t_1}^{t_2} \sum \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i \, dt = 0$

$\rightarrow \int_{t_1}^{t_2} \sum \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right]\delta q_i \, dt = 0.$

$\Rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$ (proved).

Canonical Transformation:

Let q_k and p_k be the old position and momentum co-ordinates (generalised) and Q_k and P_k are corresponding new sets. They are related by following transformation

$P_k = P_k(q_k, p_k, t)$, $Q_k = Q_k(q_k, p_k, t)$ — (1)

Hamiltonian: $H' = H'(P_k, Q_k, t)$ (new), $H = H(q_k, p_k, t)$ (old).

Hamiltonian eq^n of motion:

$\dot{P}_k = -\frac{\partial H'}{\partial Q_k}$, $\dot{Q}_k = \frac{\partial H'}{\partial P_k}$, $\dot{p}_k = -\frac{\partial H}{\partial q_k}$, $\dot{q}_k = \frac{\partial H}{\partial p_k}$ —(2)

$H' = \sum_k \dot{Q}_k P_k - L'$, $H = \sum_k \dot{q}_k p_k - L$. —(3)

Substituting L' in the Hamiltonian Principle:

$\delta \int_{t_1}^{t_2} L' dt = 0$ —(4) also, $\delta \int_{t_1}^{t_2} L\, dt = 0$.

These gives the correct eq^n of new co-ordinates P_k, Q_k and the transformation given by eq. (1) is called canonical transformation.

Generating f^n: For the canonical transformation the Lagrangian L in old old coordinates (q_k, p_k, t) and L' in the new co-ordinates (Q_k, P_k, t) must satisfy the Hamilton's principle (eq. (4))

(4) $\xrightarrow{So;}$ $\delta \int_{t_1}^{t_2} (L - L')dt = 0$ — (5)

Eq (4) can be satisfied if $\exists$ a f^n F

s.t. $L - L' = \frac{dF}{dt}$.